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We build a family of explicit one-forms on S^3 which are shown to form a new complete set of eigenmodes for the Laplace-de Rahm operator.

I. INTRODUCTION

The problem of the determination of tensorial modes for the Laplacian on spheres is important in many areas of Physics. It has been tackled in the past as part of various works in different fields. As a consequence the references on that subject may not be so easy to find. Let us summarize the specific case of vector modes on S^3. It has been considered from different point of view: Adler [1] uses an explicit embedding in a larger space, Gerlach and Sengupta [2] solve the eigenvector equation in the hyperspherical coordinates system, and Jantzen [3] makes use of the identification between S^3 and SU(2) to obtain general properties using group theoretical methods. The main results of these works (and others) has been summarized and extended by Rubin and Ordóñez [4]-[5] and also Copeland and Toms [6]. However, the vector modes do not appear in closed form in all these works. On the contrary, the specific case of transverse vector modes are given explicitly by Higuchi [7] in relation with the representations of the SO(4) group. In the present paper new vectors, or more exactly one-forms, modes for the Laplace-de Rahm operator on S^3 are built, in a compact manner, upon scalar modes using differential geometric methods.

In our work, the basic idea for finding the modes is reminiscent of that used for the vector modes for the Laplacian on $S^2 \subset \mathbb{R}^3$. That is, starting from a scalar mode one builds up its gradient ($\overrightarrow{\nabla}\Phi$), then the curl of the scalar field times a constant vector ($\overrightarrow{\nabla} \times \overrightarrow{a}\Phi$), then the curl of this vector ($\overrightarrow{\nabla} \times \overrightarrow{\nabla} \times \overrightarrow{a}\Phi$). Here the gradient will be replaced by the exterior derivative, the curl by the operator $*d$, and the constant vector $\overrightarrow{a}$ by a Killing vector of unit norm.

The structure of the paper is as follows. Notations, conventions and useful formulas are collected in Sec. I. The scalar modes in the Hopf coordinates and some of their properties are reminded in Sec. II. The main result is detailed in Sec. III.

Notations and conventions

Our notations follow those of [8]. The co-vector associated with a one-form α is denoted $\widetilde{\alpha} := \sharp\alpha$, and the one-form associated with a vector ξ is also denoted $\widetilde{\xi} := \flat\xi$. The interior product with a vector ξ is denoted by i_ξ and the exterior product with a one-form α by $j_\alpha\beta := \alpha \wedge \beta$, for any p-form β. We note that the operators i and j are nilpotent. For a p-form α we define $\hat\eta$ by $\hat\eta\alpha = (-1)^p\alpha$. The scalar product for p-forms α, β on S^3 is $\langle\, \alpha, \beta \,\rangle := \int_{S^3} \alpha \wedge *\beta$. The Laplace-de Rahm operator is defined as $\Delta := -(\delta d + d\delta)$.

For convenience we reproduce here the relations we repeatedly used in calculations on S^3, they are: $i* = *j\hat\eta$, $*\hat\eta = -\hat\eta *$, $\delta * = -*d\hat\eta$, $d* = *\delta\hat\eta$, $[\Delta, *] = 0$; for a Killing vector field ξ, one has $[\mathcal{L}_\xi, X] = 0$ for $X = *, d, \sim$; for α, β p-forms and γ a one-form one has $\langle\, j_\gamma\alpha, \beta \,\rangle = \langle\, \alpha, i_{\widetilde\gamma}\beta \,\rangle$.

II. SCALAR HARMONICS AND HOPF COORDINATES

Let us first introduce the Hopf coordinates defined on the unit sphere S^3 by

$$\begin{cases} x^1 = & \sin\alpha\cos\varphi \\ x^2 = & \sin\alpha\sin\varphi \\ x^3 = & \cos\alpha\cos\theta \\ x^4 = & \cos\alpha\sin\theta \end{cases} \tag{1}$$

with $\alpha \in [0, \pi/2]$, $\theta, \varphi \in [0, 2\pi[$. In this system the metric element on S^3 reads $ds^2 = d\alpha^2 + \cos^2\alpha\, d\theta^2 + \sin^2\alpha\, d\varphi^2$, in the coordinates basis or $ds^2 = (e^\alpha)^2 + (e^\theta)^2 + (e^\varphi)^2$ in the orthonormal, direct co-frame $e^\alpha := d\alpha$, $e^\theta := \cos\alpha\, d\theta$, $e^\varphi := \sin\alpha\, d\varphi$.

We now consider the normalized scalar modes for the Laplace-de Rahm operator $\Delta := -(d\delta + \delta d)$ on S^3 (see [9] for instance). They satisfy the eigenvalues equation

$$\Delta\Phi_i = \lambda_i\Phi_i, \tag{2}$$

where Φ_i stands for the modes corresponding to the eigenvalue $\lambda_i = -L(L+2)$, with $L \in \mathbb{N}$, the index i is a shorthand for the indexes needed to label the modes. In the system (1), for instance, the modes are labeled by

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the three numbers (L, m_+, m_-), where m_+, m_- are such that $|m_\pm| \leqslant \frac{L}{2}$, and $\frac{L}{2} - m_\pm \in \mathbb{N}$, they read

$$\begin{aligned}\Phi_i &= T_{L,m_+,m_-}(\alpha, \varphi, \theta) \qquad (3)\\ &:= C_{L,m_+,m_-} e^{i(S\varphi + D\theta)}(1-x)^{\frac{S}{2}}(1+x)^{\frac{D}{2}} P^{(S,D)}_{\frac{L}{2}-m_+}(x),\end{aligned}$$

in which $P_n^{(a,b)}$ is a Jacobi polynomial, $x := \cos 2\alpha$, $S := m_+ + m_-$, $D := m_+ - m_-$ and

$$C_{L,m_+,m_-} := \frac{1}{2^{m_+}\pi}\sqrt{\frac{L+1}{2}}\sqrt{\frac{(L/2+m_+)!(L/2-m_+)!}{(L/2+m_-)!(L/2-m_-)!}}.$$

Let us finally introduce the two Killing vectors

$$\xi := X_{12} + X_{34} = \partial_\varphi + \partial_\theta, \qquad (4)$$

$$\xi' := X_{12} - X_{34} = \partial_\varphi - \partial_\theta, \qquad (5)$$

where $X_{ij} := x_i\partial_j - x_j\partial_i$ are the generators of the so(4) algebra. Using the expression of the S^3-metric we see that $\parallel\xi\parallel = \parallel\xi'\parallel = 1$. A straightforward calculation shows that the associated one-forms to these Killing vectors are eigenvectors of the operator $*d$, one has

$$* d\widetilde{\xi} = -2\widetilde{\xi}, \quad * d\widetilde{\xi'} = +2\widetilde{\xi'}. \qquad (6)$$

In addition, the scalar modes Φ_i are eigenmodes of ξ and ξ' (seen as differential operators), one has

$$\xi(\Phi_i) = \mu_i\Phi_i, \quad \xi'(\Phi_i) = \nu_i\Phi_i,$$

where $\mu_i = +2im_+$, $\quad \nu_i = -2im_-$.

III. ONE-FORM HARMONICS

In this section, we are interested in the eigenvalue equation $\Delta\alpha = \lambda\alpha$, where α is a one-form. The space of eigenvectors for this equation is the direct sum of two orthogonal subspaces containing respectively the exact and the co-exact one-forms. The exact one-forms are given by the exterior derivatives of the scalar modes, their eigenvalues are $-L(L+2)$, $L \in \mathbb{N}\diagdown\{0\}$, the dimension of the associated proper subspace $\mathcal{E}_L^E$ is $d^E = (L+1)^2$ [6]. For the co-exact one-forms the eigenvalues are known to be $-L^2, L \in \mathbb{N}\diagdown\{0,1\}$ and the dimension of the associated proper subspaces $\mathcal{E}_L^{CE}$ is $d^{CE} = 2(L-1)(L+1)$ [6]. Here, we will build up an explicit new orthonormal basis of co-exact eigenmodes. Our strategy will be to exhibit a family of modes associated to the eigenvalue $-L^2$, to show their orthogonality, and finally to check that their number is precisely the dimension of the proper subspace $\mathcal{E}_L^{CE}$.

A. Definition of the modes

Using the scalar modes let us define:

$$\begin{aligned}A_i &:= d\Phi_i, & &\\ B_i &:= *d\Phi_i\widetilde{\xi}, & B'_i &:= *d\Phi_i\widetilde{\xi'},\\ C_i &:= *dB_i, & C'_i &:= *dB'_i,\end{aligned}$$

in which A_i is an exact one-form while B_i, C_i, B'_i and C'_i are co-exact one-forms. Let us, in addition, consider the combinations

$$E_i := (L+2)B_i + C_i, \quad E'_i := (L+2)B'_i - C'_i, \qquad (7)$$

where $i = (L, m_+, m_-)$ (respectively (L, m'_+, m'_-)).

B. Statement of the results

The following properties hold:

1. The one-forms E_{L,m_+,m_-} and E'_{L,m'_+,m'_-} satisfy

$$\Delta E_{L,m_+,m_-} = -L^2 E_{L,m_+,m_-},$$
$$\Delta E'_{L,m'_+,m'_-} = -L^2 E'_{L,m'_+,m'_-},$$

for $L \geqslant 2$.

2. The family of one-forms

$$\begin{cases} E_{L,m_+,m_-}, L \geqslant 2, |m_+| \leqslant \frac{L}{2} - 1, |m_-| \leqslant \frac{L}{2},\\ E'_{L,m'_+,m'_-}, L \geqslant 2, |m'_-| \leqslant \frac{L}{2} - 1, |m'_+| \leqslant \frac{L}{2},\end{cases}$$

once normalized, form an orthonormal basis of the corresponding proper subspace of co-exact one-forms $\mathcal{E}_L^{CE}$.

3. The whole set of one-forms: exact $\{A_{L,m_+,m_-}\}$, for $L \geqslant 1$, and co-exact $\{E_{L,m_+,m_-}, E'_{L,m'_+,m'_-}\}$, as above, form a complete orthonormal set of modes for the Laplace-de Rahm operator on S^3.

Finally, let us note that the modes E_i and E'_i can be recast under a vectorial form which is reminiscent of the results for the two-sphere reminded in the introduction Sec. I, namely

$$\overrightarrow{E}_i = (L+2)\left(\overrightarrow{\nabla}\times(\Phi_i\overrightarrow{\xi})\right) + \overrightarrow{\nabla}\times\overrightarrow{\nabla}\times(\Phi_i\overrightarrow{\xi}),$$
$$\overrightarrow{E}'_i = (L+2)\left(\overrightarrow{\nabla}\times(\Phi_i\overrightarrow{\xi}')\right) - \overrightarrow{\nabla}\times\overrightarrow{\nabla}\times(\Phi_i\overrightarrow{\xi}').$$

C. Proof of the first property

We first note the following property: if a co-exact one-form α is an eigenmode of $*d$ with the eigenvalue σ then α is an eigenmode of Δ with the eigenvalue σ^2. Indeed, one has:

$$\Delta\alpha = -\delta d\alpha = - * d * \hat{\eta} d\alpha = - * d * d\alpha = -\sigma^2\alpha.$$

We consequently first consider the operator $*d$.

The action of $*d$ on B_i is just the definition of C_i (Sec. III A). It remains to determine $*dC_i$. From the definition of B_i and C_i one has, using (6) and $\Delta\Phi_i = -\delta d\Phi_i$,

$$\begin{aligned}
C_i =& * dB_i \\
=& * d * dj_{\tilde{\xi}}\Phi_i \\
=& - *d * j_{\tilde{\xi}} d\Phi_i + *d\Phi_i * d\tilde{\xi} \\
=& * di_\xi * d\Phi_i + *d\Phi_i(-2)\tilde{\xi} \\
=& * (\mathcal{L}_\xi - i_\xi d) * d\Phi_i - 2B_i \\
=& d\mathcal{L}_\xi \Phi_i - *i_\xi d * d\Phi_i - 2B_i \\
=& d\mu_i\Phi_i + j_{\tilde{\xi}}\delta d\Phi_i - 2B_i \\
C_i =& \mu_i A_i - \lambda_i\Phi_i\tilde{\xi} - 2B_i.
\end{aligned} \tag{8}$$

Applying $*d$ to the above expression of C_i, taking into account the exactness of A_i, leads to

$$* dC_i = -\lambda_i B_i - 2C_i. \tag{9}$$

This relation together with the definition of C_i, namely $*dB_i =: C_i$, is a closed system of equations which can be diagonalized to obtain eigenmodes of $*d$. A straightforward calculation with λ_i replaced by its value $-L(L+2)$ leads to

$$*dE_i = + LE_i, \tag{10}$$

$$*dF_i = - (L+2)F_i. \tag{11}$$

where $E_i := (L+2)B_i + C_i$ is the combination given in Sec III A and $F_i := LB_i - C_i$ is another set of modes that we do not need to consider further. We then apply the property quoted at the beginning of this section to the co-exact one-form E_i, this leads to the result

$$\Delta E_i = -L^2 E_i. \tag{12}$$

A completely analogous calculation using B'_i and C'_i in place of B_i and C_i shows that the E'_i's are eigenmodes of $*d$ with opposite eigenvalues and thus of Δ with the same eigenvalues. This completes the proof of the first property.

We thus have two families of modes with completely similar properties, the formulas for the E'_i's being obtained through the same calculations, but using primed quantities (B'_i, C'_i, ...), as those leading to the results for the E_i's . Consequently we will consider the family $\{E_i\}$ and only state the results for the family $\{E'_i\}$.

The formula (12) is valid for any $L \geqslant 0$, nevertheless the eigenvalue $L^2 = 0$ is excluded because it corresponds to harmonic one-forms whose set on a Riemannian manifold with positive Ricci curvature is known to only contains the null one-form (Bochner's theorem). As we will see in the forthcoming paragraph this is in accordance with the norm of the corresponding eigenvector E_{000} ($L = 0$) which vanishes. Moreover, we will see that E_i and E'_i also vanish for $L = 1$. This explains the additional condition $L \geqslant 2$ for the eigenvalues of the modes E_i and E'_i.

D. Scalar products and norms

From their definition (7) the scalar products between the E_i's and E'_i's can be deduced from those between the B_i's, C_i's, B'_i's and C'_i's, which we compute hereafter.

We begin by the scalar product between the B_i's which reads

$$\begin{aligned}
\langle B_i, B_j \rangle =& \langle *dj_{\tilde{\xi}}\Phi_i, *dj_{\tilde{\xi}}\Phi_j \rangle \\
=& \langle \Phi_i, i_\xi \delta dj_{\tilde{\xi}}\Phi_j \rangle.
\end{aligned}$$

In order to calculate the term in the bracket we observe that $\delta dj_{\tilde{\xi}}\Phi_j = C_j$, using the expression (8) we now calculate $i_\xi C_j$, one has

$$\begin{aligned}
i_\xi C_j =& i_\xi(\mu_j d\Phi_j - \lambda_j\Phi_j\tilde{\xi} - 2 * dj_{\tilde{\xi}}\Phi_j) \\
=& \mu_j(\mathcal{L}_\xi - di_\xi)\Phi_j - \lambda_j\Phi_j \parallel\xi\parallel^2 \\
& -2(i_\xi\Phi_j * d\tilde{\xi} - i_\xi * j_{\tilde{\xi}} d\Phi_j) \\
=& \mu_j\xi(\Phi_j) - \lambda_j\Phi_j \parallel\xi\parallel^2 + 4\Phi_j \parallel\xi\parallel^2 \\
=& (\mu_j^2 - \lambda_j + 4)\Phi_j,
\end{aligned}$$

where we used $i_\xi * j_{\tilde{\xi}} = 0$ and $\parallel\xi\parallel = 1$. Finally,

$$\langle B_i, B_j \rangle = (\mu_i^2 - \lambda_i + 4)\delta_{ij}, \tag{13}$$

where δ_{ij} has to be interpreted as the product of the Kronecker symbols of the various numbers labeling the modes. For $i = (L, m_+, m_-)$ and $j = (K, n_+, n_-)$ one has $\delta_{ij} = \delta_{LK}\delta_{m_+n_+}\delta_{m_-n_-}$.

The product $\langle B_i, C_j \rangle$ reads

$$\begin{aligned}
\langle B_i, C_j \rangle =& \langle *dj_{\tilde{\xi}}\Phi_i, C_j \rangle \\
=& \langle \Phi_i, i_\xi * dC_j \rangle.
\end{aligned}$$

Keeping in mind the previous calculations, the r.h.s. of the bracket reads

$$\begin{aligned}
i_\xi * dC_j =& i_\xi * d(\mu_j A_j - \lambda_j\Phi_j\tilde{\xi} - 2B_j) \\
=& - \lambda_j i_\xi * d\Phi_j\tilde{\xi} - 2i_\xi C_j \\
=& - \lambda_j i_\xi(- * j_{\tilde{\xi}} d\Phi_j + \Phi_j * d\xi) \\
& -2i_\xi(\mu_j A_j - \lambda_j\Phi_j\tilde{\xi} - 2 * dj_{\tilde{\xi}}\Phi_j) \\
=& - \lambda_j i_\xi(i_\xi * d\Phi_j + \Phi_j(-2\tilde{\xi})) \\
& -2(\mu_j^2\Phi_j - \lambda_j \parallel\xi\parallel^2 \Phi_j - 2i_\xi * dj_{\tilde{\xi}}\Phi_j) \\
=& - 2(\mu_j^2 - 2\lambda_j \parallel\xi\parallel^2 + 4 \parallel\xi\parallel^2)\Phi_j.
\end{aligned}$$

Finally, with $\parallel\xi\parallel = 1$ we obtain

$$\langle B_i, C_j \rangle = -2(\mu_i^2 - 2\lambda_i + 4)\delta_{ij}. \tag{14}$$

The product between the C_i's can be calculated using the results (13) and (14), one has

$$\begin{aligned}
\langle C_i, C_j \rangle =& \langle *dB_i, *dB_j \rangle \\
=& \langle B_i, \delta dB_j \rangle \\
=& - \langle B_i, \Delta B_j \rangle \\
=& - \lambda_j \langle B_i, B_j \rangle - 2\langle B_i, C_j \rangle,
\end{aligned}$$

from which we obtain

$$\langle C_i, C_j \rangle = \left[(4-\lambda_i)\mu_i^2 + (\lambda_i - 12)\lambda_i + 16\right] \delta_{ij}. \quad (15)$$

From the scalar products (13-15) and the definition of the modes (7) a straightforward calculation leads to

$$\langle E_i, E_j \rangle = \| E_i \|^2 \delta_{ij},$$

with here $i = (L, m_+, m_-)$ and in which the squared norm is given by:

$$\| E_i \|^2 = 2L(L+1)(L^2 - 4m_+^2), \quad (16)$$

where we have used the values of $\lambda_i = -L(L+2)$ and $\mu_i = 2\, i\, m_+$.

The norm of the eigenvector E_{L,m_+,m_-} (16), vanishes for $m_+ = \pm\frac{L}{2}$, these values, which correspond to the boundaries of the spectrum, are thus excluded. For the value $L = 1$, the two values allowed for m_+ are on the boundaries, the value $L = 1$ is thus excluded.

Finally, a completely analogous calculation using the primed quantities (B_i', C_i',...) leads to the same results as above for the modes E_i' in which μ_i is replaced by ν_i that is m_+ is replaced by m_- in the squared norm of E_i'.

E. Proof of the second property

The dimension of the proper subspace associated to a given eigenvalue is given by its degeneracy. Following [6] (appendix B) the degeneracy of the eigenvalue $-L^2$ is $d^{CE} = 2(L-1)(L+1)$. Now, the results of Sec. III D show that for a given eigenvalue the number of eigenvectors E_i is given by the number of values for m_+ which correspond to a non-null eigenvector times the number of possible values for m_-. Then, for the eigenvalue $-L^2$ the number of E_{L,m_+,m_-} eigenvectors is $(L-1)(L+1)$. This is also the number of eigenvectors E'_{L,m'_+,m'_-} for the same eigenvalue. Consequently, to prove the second property it is sufficient to show that the two families E_i and E_i' are orthogonal. This is a consequence of the fact that the $*d$ operator is symmetric, namely

$$\begin{aligned}\langle *dE_i, E_j' \rangle =& \langle E_i, \delta * E_j' \rangle \\ =& \langle E_i, *d * \hat{\eta} * E_j' \rangle \\ =& \langle E_i, *d * *E_j' \rangle \\ =& \langle E_i, *dE_j' \rangle.\end{aligned}$$

It follows that two eigenmodes corresponding to two different eigenvalues of $*d$ are orthogonal. Now, as quoted in the Sec. III C the eigenvalues corresponding to E_i and E_i' are respectively positive and negative integers and consequently have no value in common. Thus the modes E_i and E_j' are orthogonal. This completes the proof of the second property.

F. Proof of the third property

The proof of the third point is as follows. First note that the A_i's are eigenmodes for the Laplace-de Rahm operator

$$\begin{aligned}\Delta A_i =& -(d\delta + \delta d)d\Phi_i \\ =& -d(\delta d + d\delta)\Phi_i \\ =& \lambda_i A_i.\end{aligned}$$

Then, we determine the scalar product of two A_i's

$$\begin{aligned}\langle A_i, A_j \rangle =& \langle \Phi_i, \delta d\Phi_j \rangle \\ =& \langle \Phi_i, -\lambda_j \Phi_j \rangle \\ =& -\lambda_i \delta_{ij}.\end{aligned}$$

The family A_i is thus a complete set of exact eigenmodes on S^3 since the dimension of each proper subspace is $(L+1)^2$.

Finally, the proof of the last property amounts to show that the two sets of modes $\{A_{L,m_+,m_-}\}$ and $\{E_{L,m_+,m_-}, E'_{L,m'_+,m'_-}\}$ are orthogonal. This comes from the fact that the members of the second family are co-exact, in fact for a co-exact form α one has:

$$\langle A_i, \alpha \rangle = \langle \Phi_i, \delta\alpha \rangle = 0.$$

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